

# The Decomposition in Block Terms

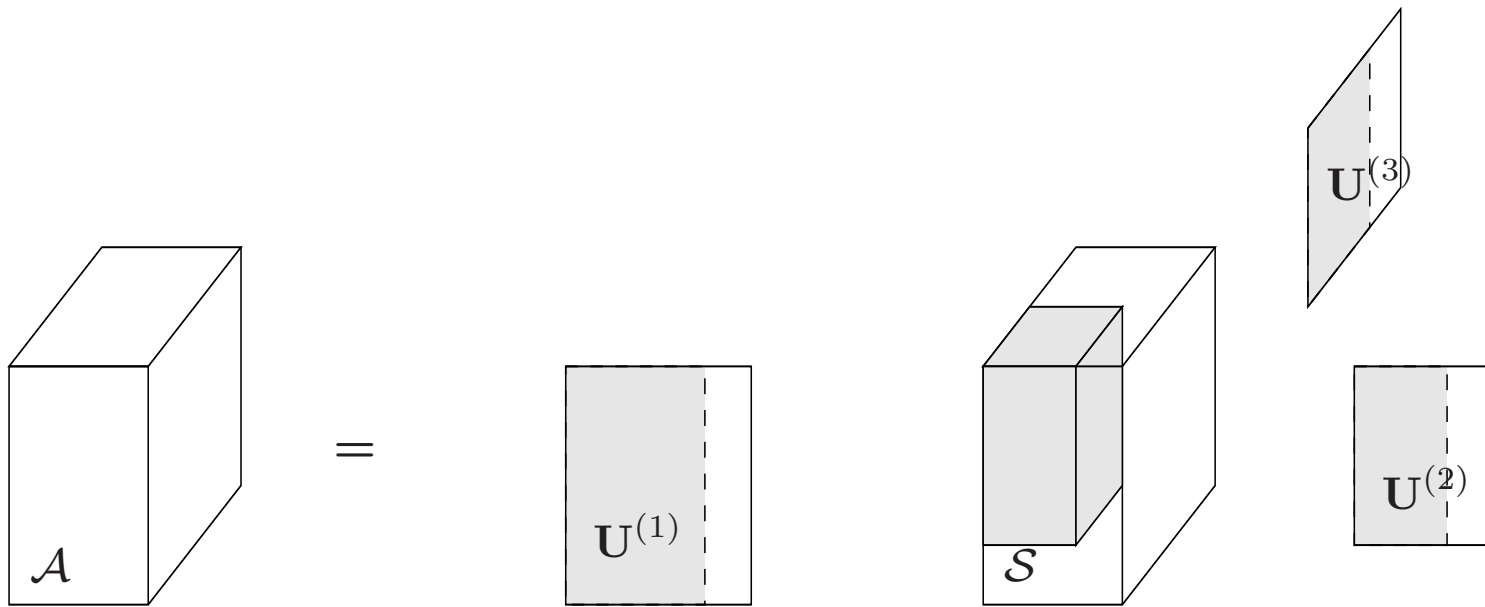
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## Tucker's decomposition and best rank- $(R_1, R_2, R_3)$ approximation



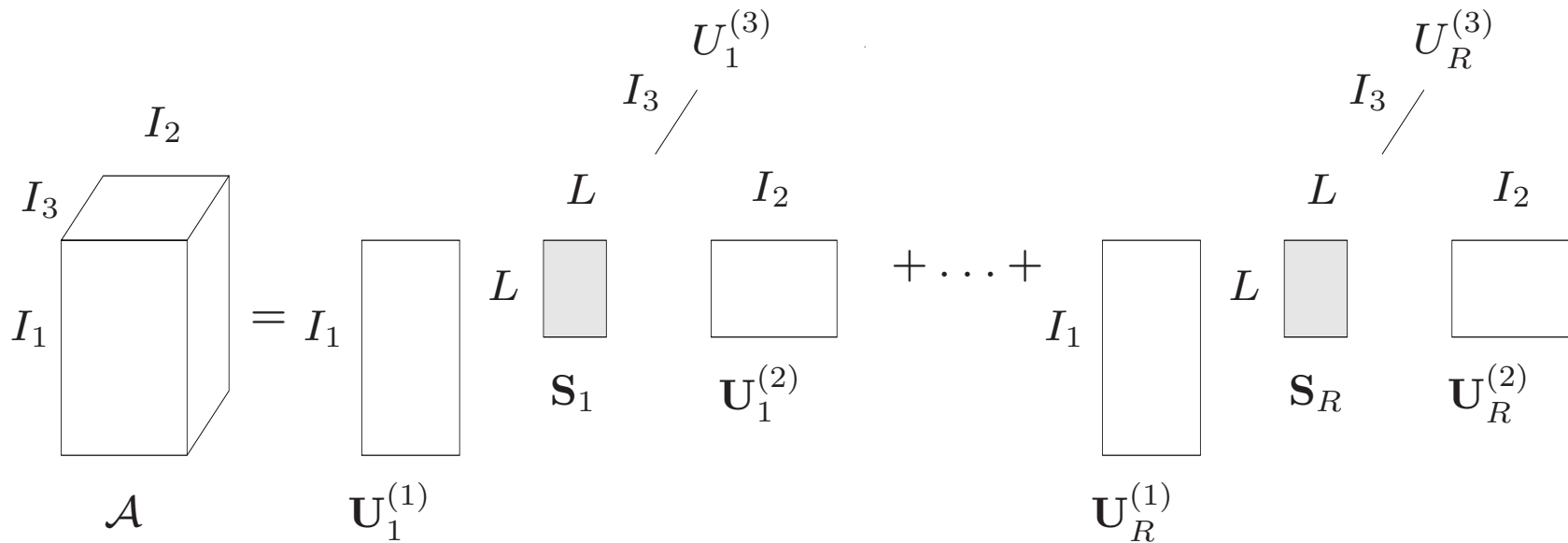
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**PARAFAC**

The diagram illustrates the PARAFAC decomposition of a 3D tensor  $A$ . On the left, a 3D box labeled  $A$  is shown. To its right is an equals sign, followed by a sum of rank-1 tensors. Each rank-1 tensor is represented by a vertical line on the left, a horizontal line on the bottom, and a diagonal line on the right. The vertical line is labeled  $U_1^{(1)}$ ,  $U_2^{(1)}$ , or  $U_R^{(1)}$  at the bottom. The horizontal line is labeled  $U_1^{(2)}$ ,  $U_2^{(2)}$ , or  $U_R^{(2)}$  on the right. The diagonal line is labeled  $U_1^{(3)}$ ,  $U_2^{(3)}$ , or  $U_R^{(3)}$  at the top. The terms are separated by plus signs, with an ellipsis between the second and third terms.

$$A = U_1^{(1)} U_1^{(2)} U_1^{(3)} + U_2^{(1)} U_2^{(2)} U_2^{(3)} + \dots + U_R^{(1)} U_R^{(2)} U_R^{(3)}$$

## Decomposition in rank- $(L, L, 1)$ terms

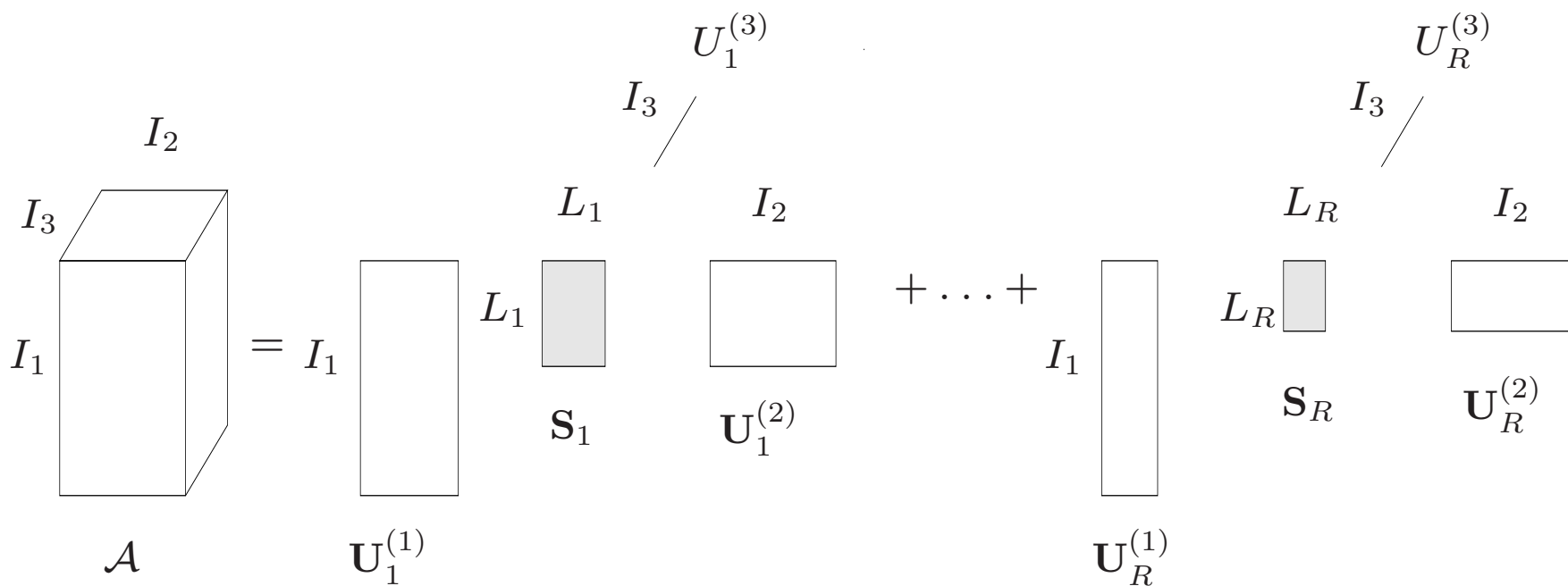


### Uniqueness

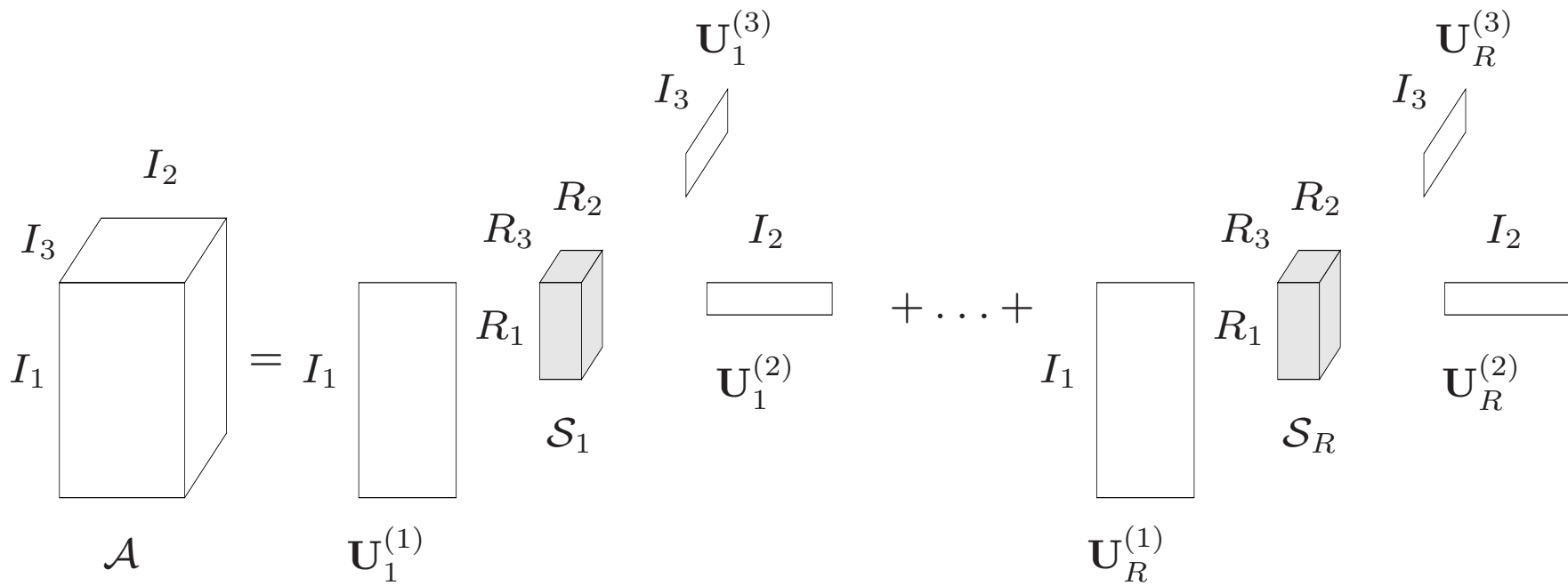
$$\min\left(\left\lfloor \frac{I_1}{L} \right\rfloor, R\right) + \min\left(\left\lfloor \frac{I_2}{L} \right\rfloor, R\right) + \min(I_3, R) \geq 2R + 2$$

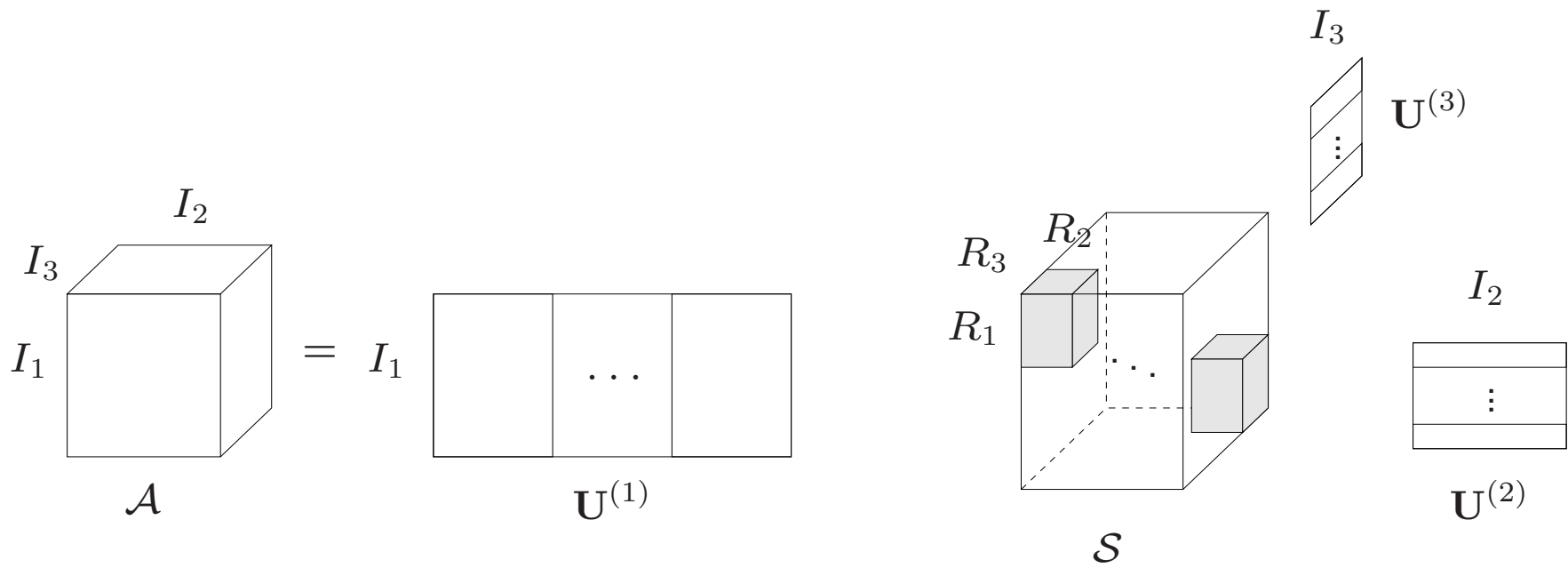
$$\text{cf. } \min(I_1, R) + \min(I_2, R) + \min(I_3, R) \geq 2R + 2 \quad (\text{PARAFAC})$$

## Decomposition in rank- $(L_r, L_r, 1)$ terms



## Decomposition in rank- $(R_1, R_2, R_3)$ terms





## Decomposition in rank-(2, 2, 2) terms

Not unique:

$$\begin{aligned}\mathcal{A} &= \mathcal{S}_1 + \mathcal{S}_2 \\ &= (U_1 \circ V_1 \circ W_1 + U_2 \circ V_2 \circ W_2) + (U_3 \circ V_3 \circ W_3 + U_4 \circ V_4 \circ W_4) \\ &= (U_1 \circ V_1 \circ W_1 + U_3 \circ V_3 \circ W_3) + (U_2 \circ V_2 \circ W_2 + U_4 \circ V_4 \circ W_4) \\ &= \tilde{\mathcal{S}}_1 + \tilde{\mathcal{S}}_2\end{aligned}$$

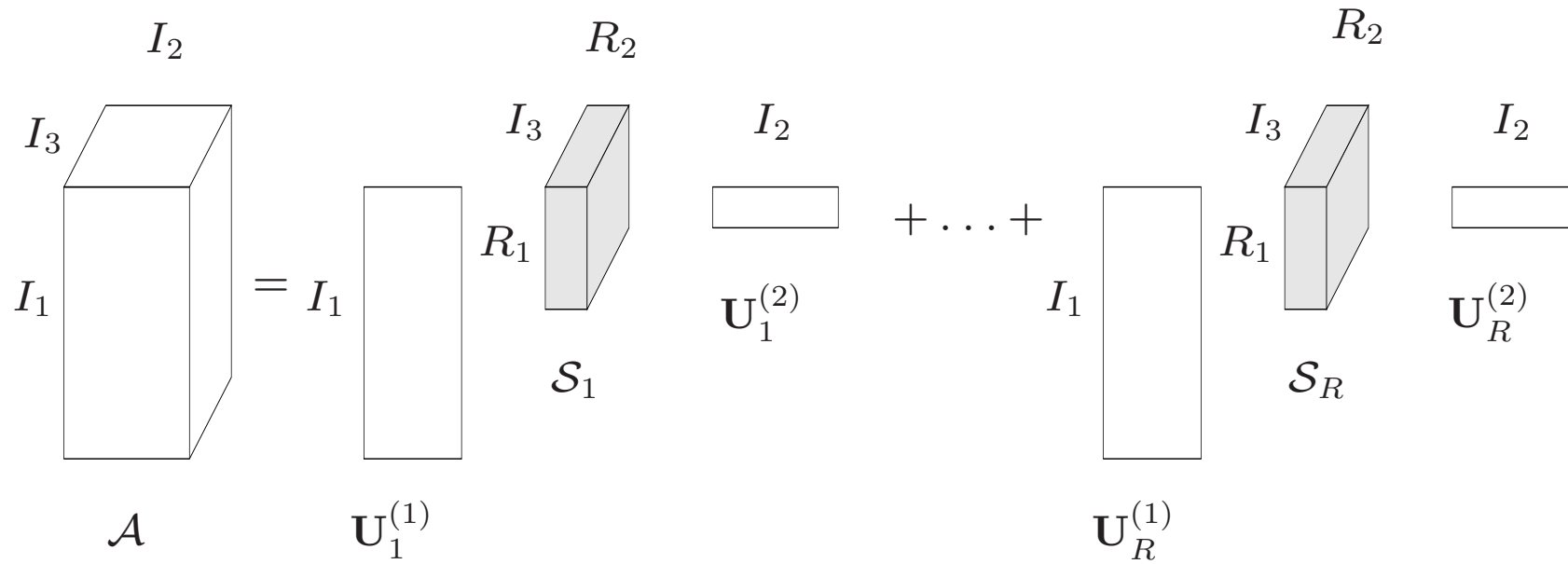
Decomposition in rank-(2, 2, 3) terms:

unique if

$$\left\lfloor \frac{I_1}{2} \right\rfloor + \left\lfloor \frac{I_2}{2} \right\rfloor + \left\lfloor \frac{I_3}{3} \right\rfloor \geq 2R + 2$$



## Type-2 decomposition in rank- $(R_1, R_2, \cdot)$ terms



## **Block Factor Analysis, a new concept for signal separation**

## A decomposition structure

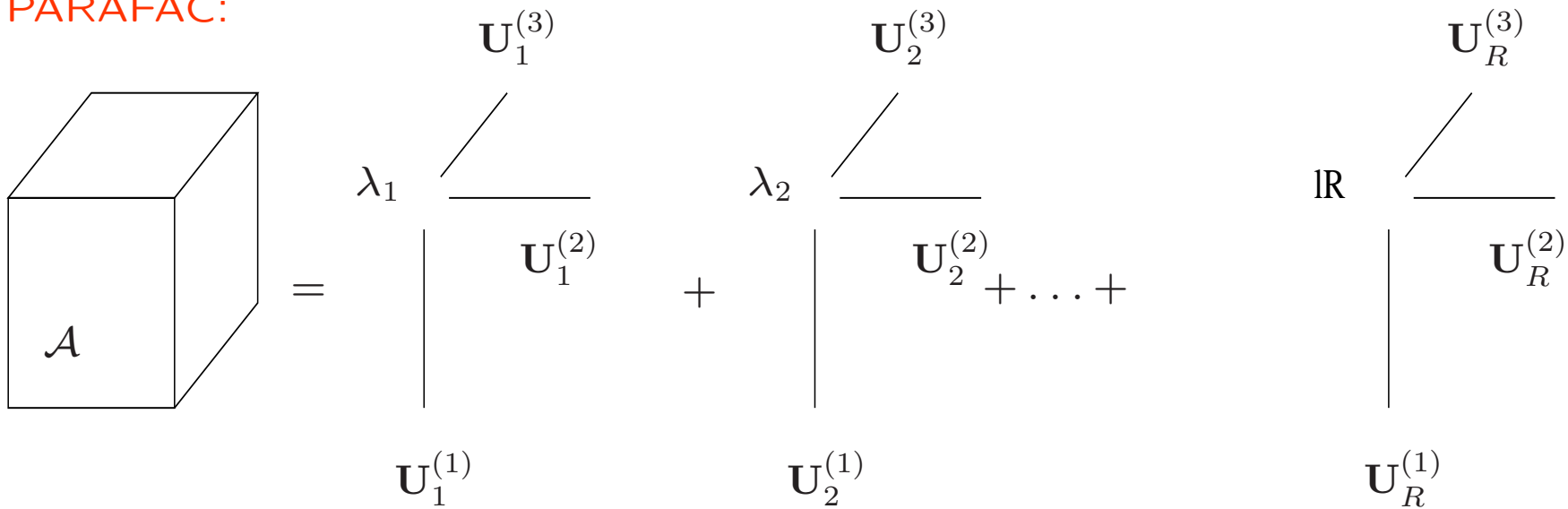
$(I_1 \times I_2 \times I_3)$  tensor  $\mathcal{A}$ :

$$\left| \begin{array}{c} (I_1, I_2, I_3) \\ (I_1, I_2, I_3 - 1) \\ (I_1 - 1, I_2 - 2, I_3 - 3) \\ \vdots \\ 1 \end{array} \right| \left| \begin{array}{c} 2(I_1, I_2, \lceil I_3/2 \rceil) \\ 2(I_1, I_2, \lfloor I_3/2 \rfloor) \\ (I_1, I_2, \lfloor I_3/2 \rfloor) + (I_1, \lfloor I_2/2 \rfloor, I_3) \\ \vdots \\ 2(1) \end{array} \right| \cdots \left| \begin{array}{c} R(1) \end{array} \right|$$

- rank
- generalization SVD
- typical rank
- degeneracy
- complex factors cf. [Kaporin, '05]

## ALS algorithm

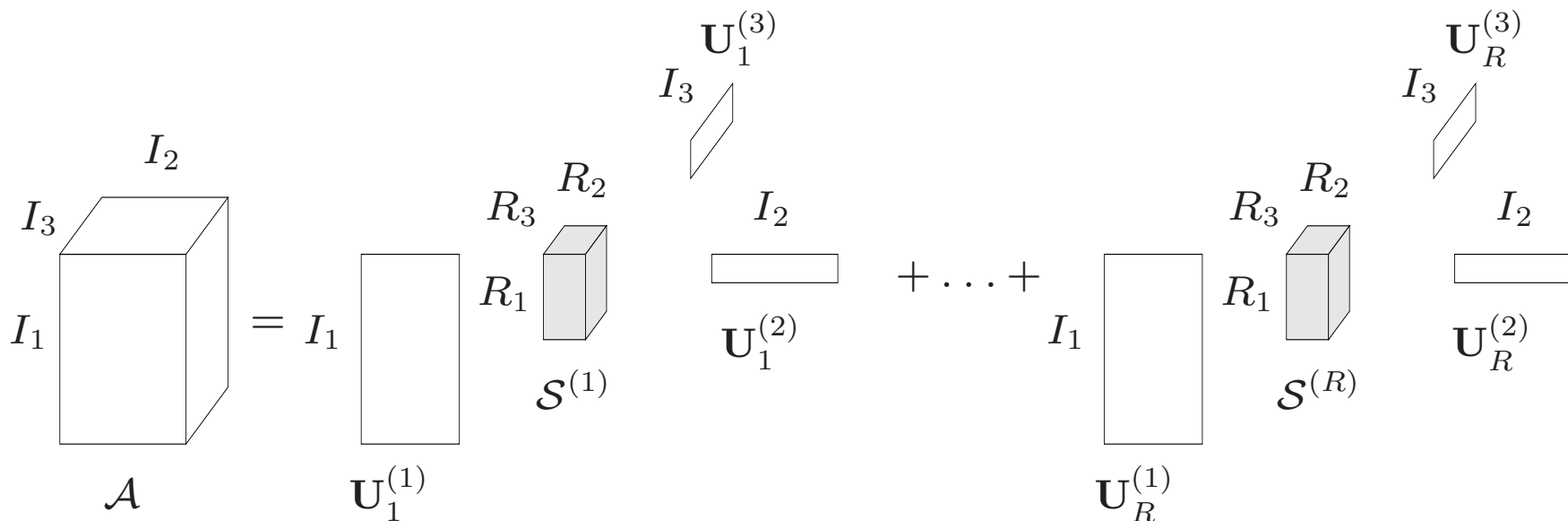
PARAFAC:



$$\mathbf{M}(\{U_r^{(2)}\}, \{U_r^{(3)}\})X = A$$

$$x(1 : I_1) \sim \lambda_1 U_1^{(1)}$$

Decomposition in rank- $(R_1, R_2, R_3)$  terms:



$$\mathbf{M}(\{\mathbf{U}_r^{(2)}\}, \{\mathbf{U}_r^{(3)}\}) \mathbf{X} = A$$

$$x(1 : I_1 R_2 R_3) \sim \mathbf{U}_1^{(1)} \cdot \mathbf{S}^{(1)} \\ (I_1 \times R_1)(R_1 \times R_2 R_3)$$

## Perspectives

- Simultaneous matrix decompositions
- Enhanced line search
- Levenberg-Marquardt
- Model order and model structure selection