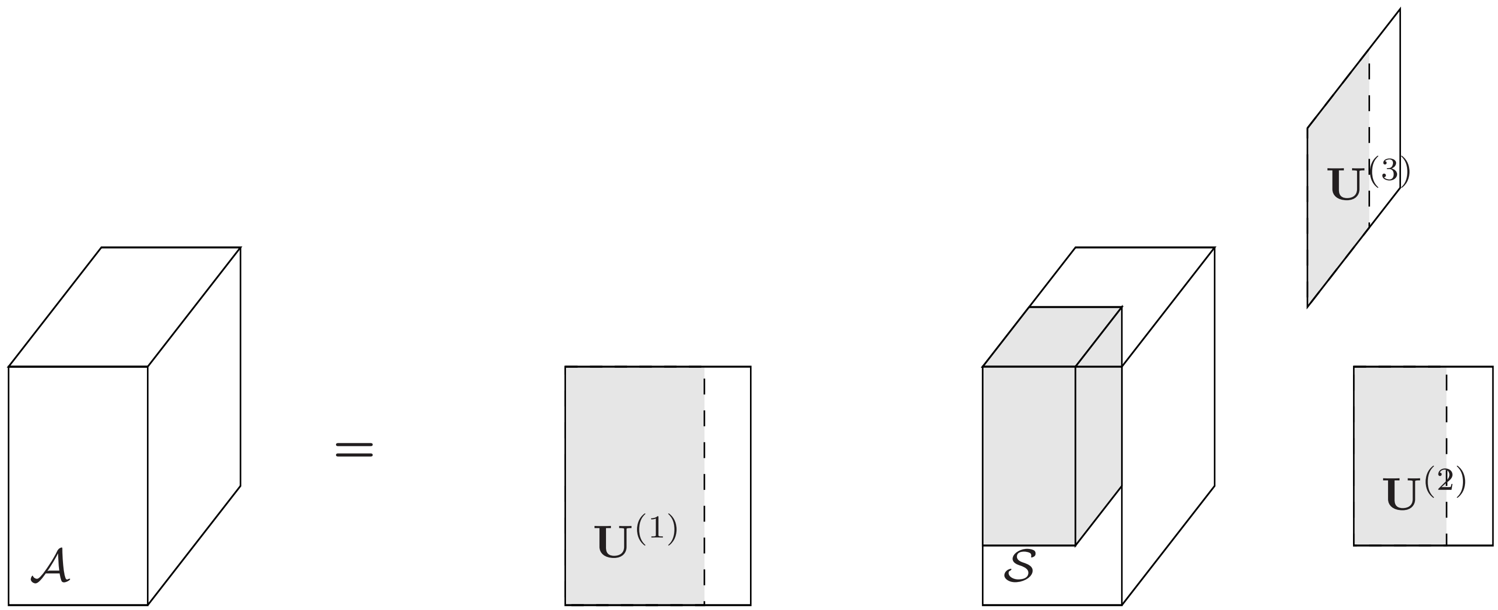
The Decomposition in Block Terms

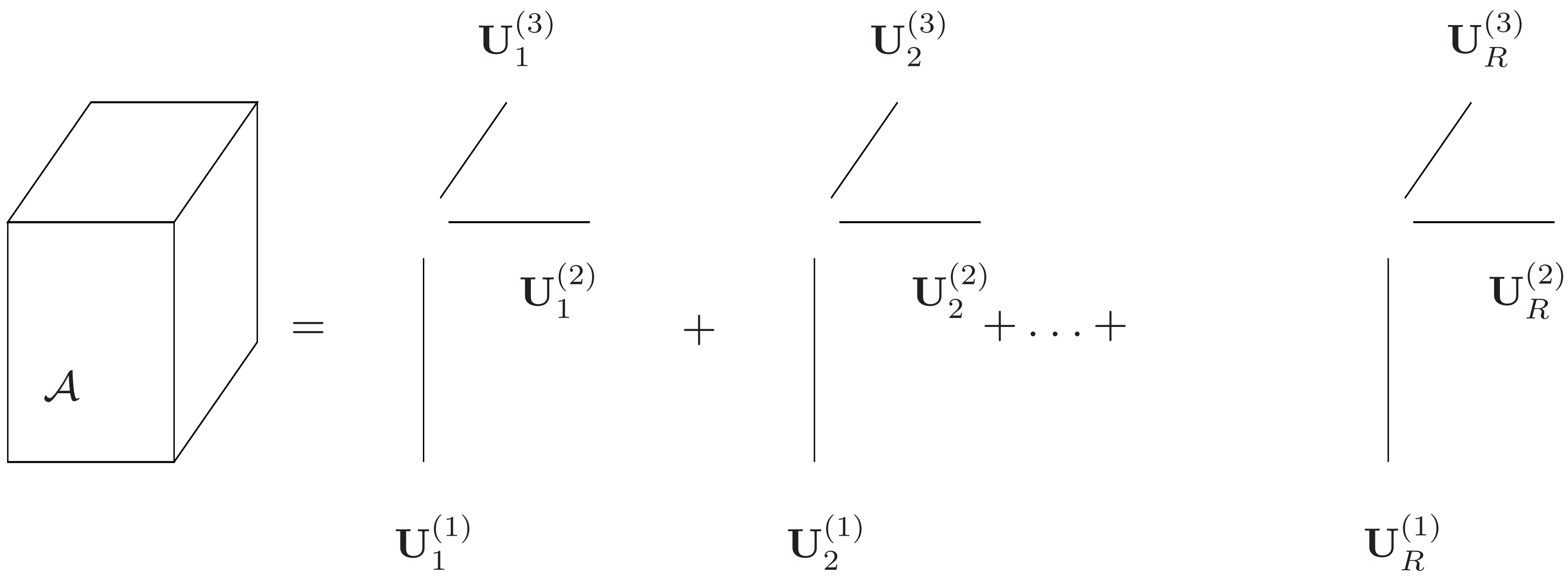
Lieven De Lathauwer

CNRS ETIS - Cergy-Pontoise
France
delathau@ensea.fr

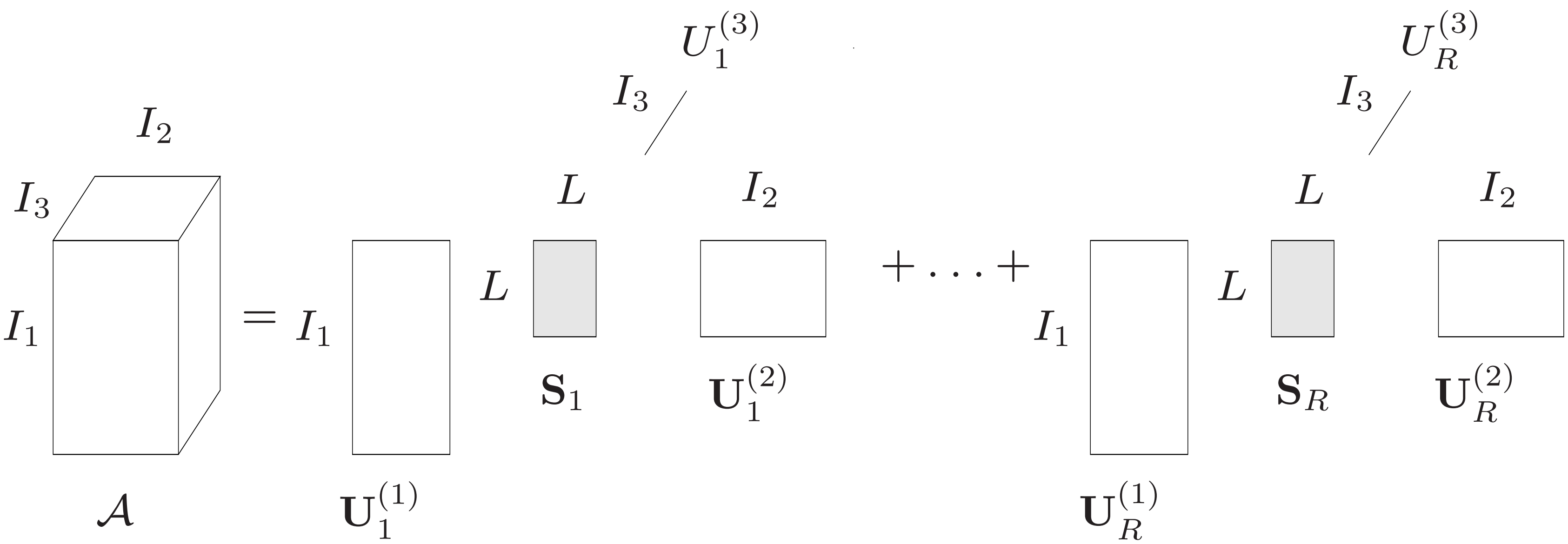
Tucker's decomposition and best rank-(R_1, R_2, R_3) approximation



PARAFAC


$\mathcal{A}$
=
$\mathbf{U}_1^{(3)}$
$\mathbf{U}_1^{(2)}$
$\mathbf{U}_1^{(1)}$
+
$\mathbf{U}_2^{(3)}$
$\mathbf{U}_2^{(2)}$
$\mathbf{U}_2^{(1)}$
$+ \ldots +$
$\mathbf{U}_R^{(3)}$
$\mathbf{U}_R^{(2)}$
$\mathbf{U}_R^{(1)}$

Decomposition in rank-$(L, L, 1)$ terms

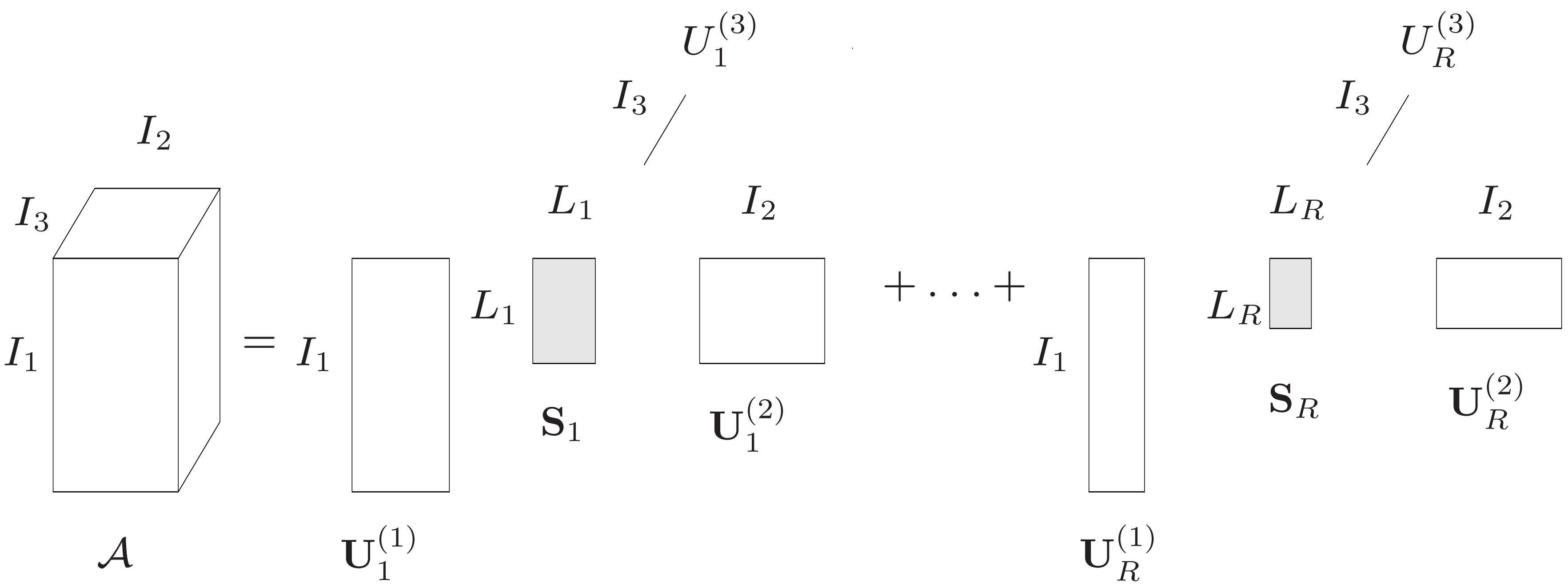


Uniqueness

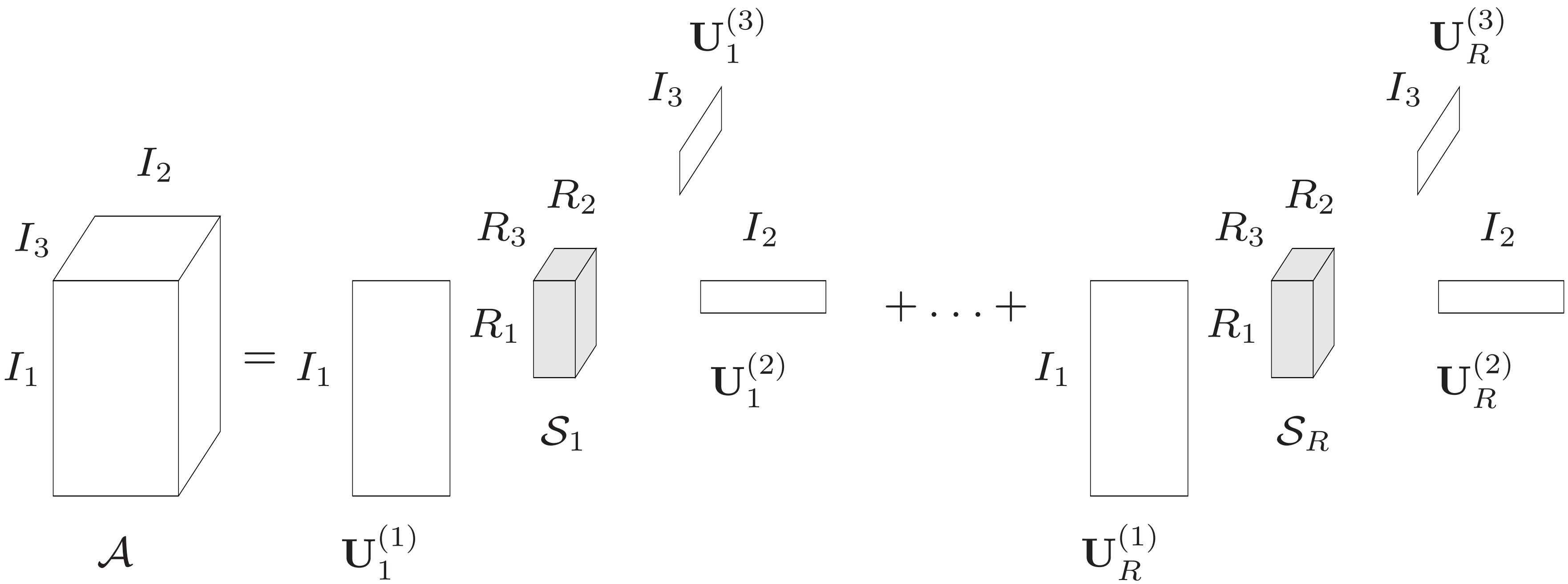
$$\min\left(\left\lfloor \frac{I_1}{L} \right\rfloor, R\right) + \min\left(\left\lfloor \frac{I_2}{L} \right\rfloor, R\right) + \min(I_3, R) \quad \geqslant \quad 2R + 2$$

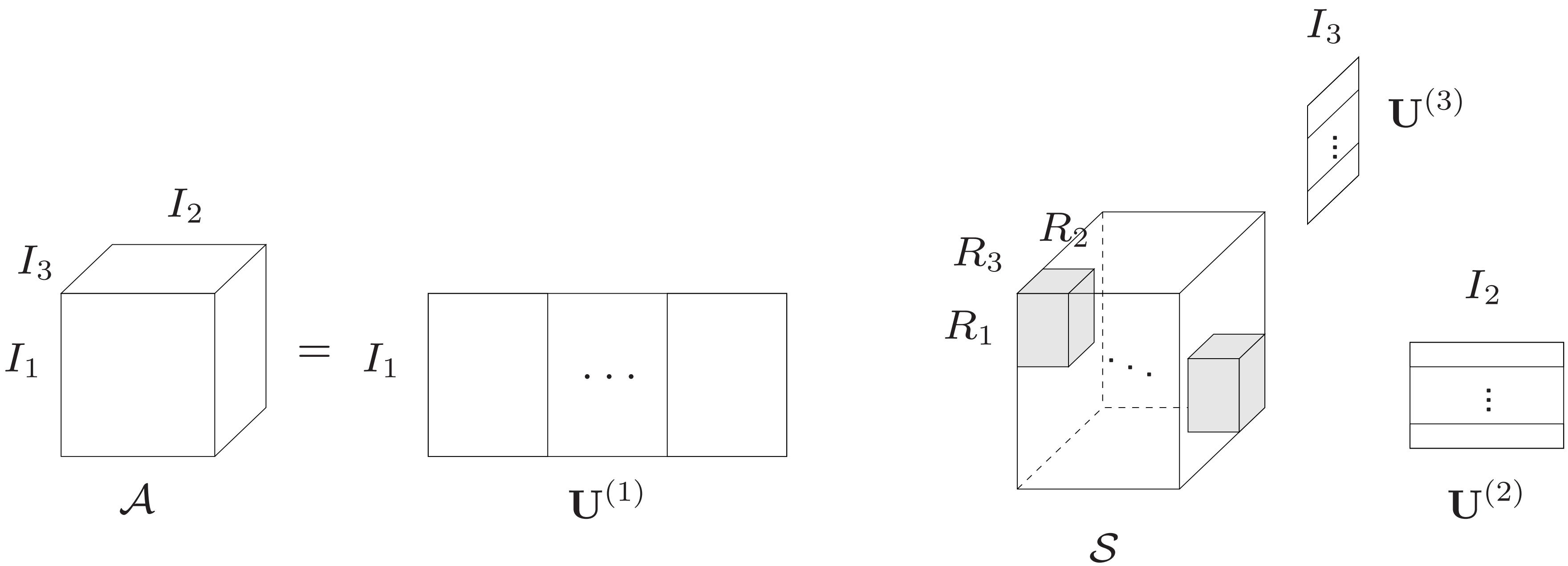
cf. $\quad \min(I_1, R) + \min(I_2, R) + \min(I_3, R) \quad \geqslant \quad 2R + 2 \quad$ (PARAFAC)

Decomposition in rank-$(L_r, L_r, 1)$ terms



Decomposition in rank-(R_1, R_2, R_3) terms




I_2
I_3
I_1
$\mathcal{A}$
$=$
I_1
$\dots$
$\mathbf{U}^{(1)}$
R_3
R_2
R_1
$\mathcal{S}$
I_3
$\mathbf{U}^{(3)}$
I_2
$\mathbf{U}^{(2)}$

Decomposition in rank-$(2, 2, 2)$ terms

Not unique:

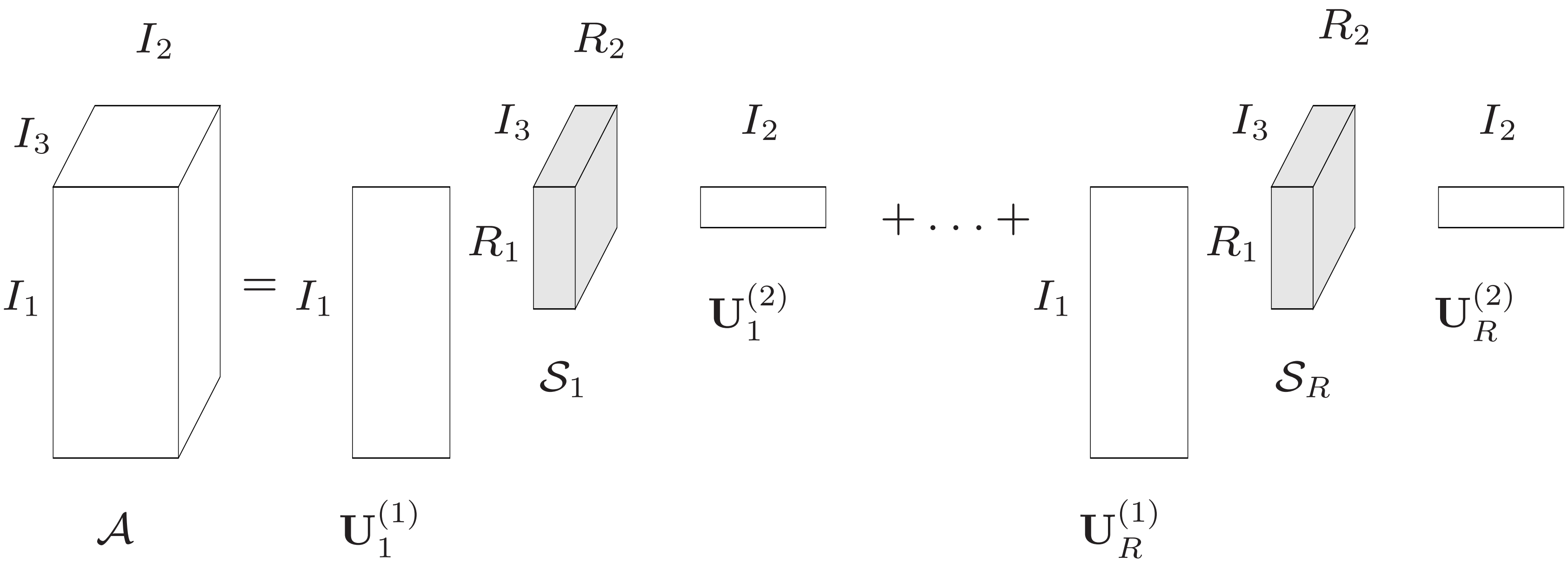
$$
\begin{aligned}
\mathcal{A} &= \mathcal{S}_1 + \mathcal{S}_2 \\
&= (U_1 \circ V_1 \circ W_1 + U_2 \circ V_2 \circ W_2) + (U_3 \circ V_3 \circ W_3 + U_4 \circ V_4 \circ W_4) \\
&= (U_1 \circ V_1 \circ W_1 + U_3 \circ V_3 \circ W_3) + (U_2 \circ V_2 \circ W_2 + U_4 \circ V_4 \circ W_4) \\
&= \tilde{\mathcal{S}}_1 + \tilde{\mathcal{S}}_2
\end{aligned}
$$

Decomposition in rank-$(2, 2, 3)$ terms:

unique if

$$\left\lfloor \frac{I_1}{2} \right\rfloor + \left\lfloor \frac{I_2}{2} \right\rfloor + \left\lfloor \frac{I_3}{3} \right\rfloor \geqslant 2R + 2$$

Type-2 decomposition in rank-$(R_1, R_2, \cdot)$ terms



Block Factor Analysis, a new concept for signal separation

A decomposition structure

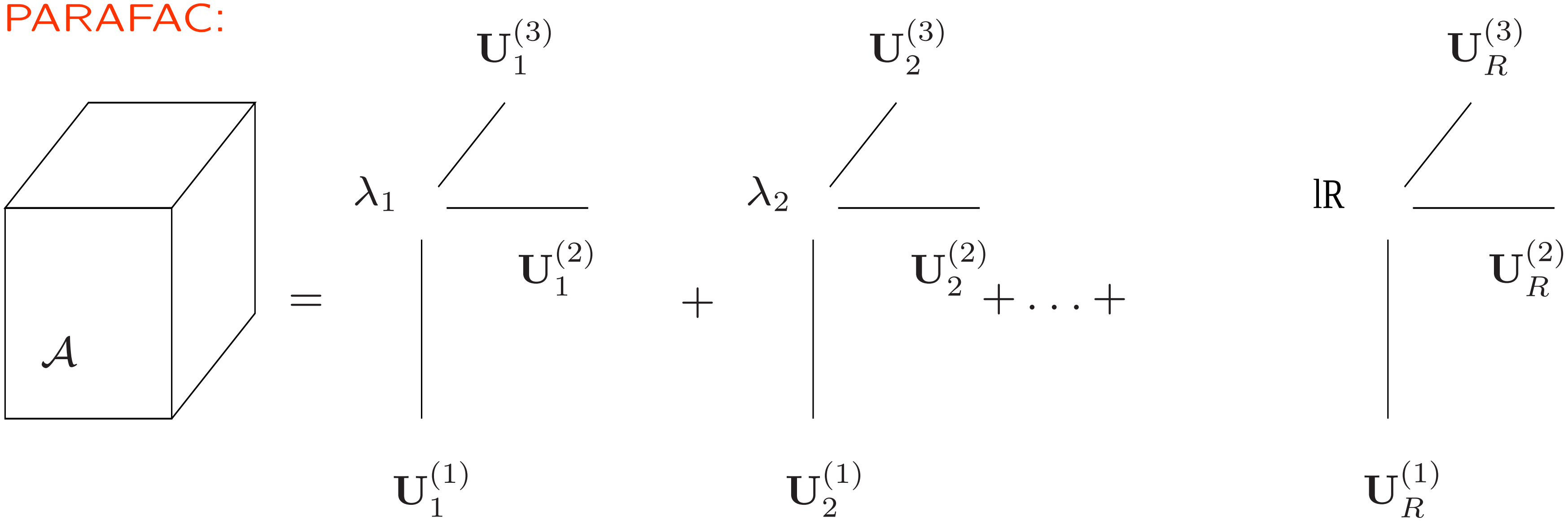
$(I_1 \times I_2 \times I_3)$ tensor $\mathcal{A}$:

(I_1, I_2, I_3)	$2(I_1, I_2, \lceil I_3/2 \rceil)$		$R(1)$
$(I_1, I_2, I_3 - 1)$	$2(I_1, I_2, \lfloor I_3/2 \rfloor)$		
$(I_1 - 1, I_2 - 2, I_3 - 3)$	$(I_1, I_2, \lfloor I_3/2 \rfloor) + (I_1, \lfloor I_2/2 \rfloor, I_3)$	$\cdots$	
$\vdots$	$\vdots$		
1	$2(1)$		

- rank
- generalization SVD
- typical rank
- degeneracy
- complex factors cf. [*Kaporin, '05*]

ALS algorithm

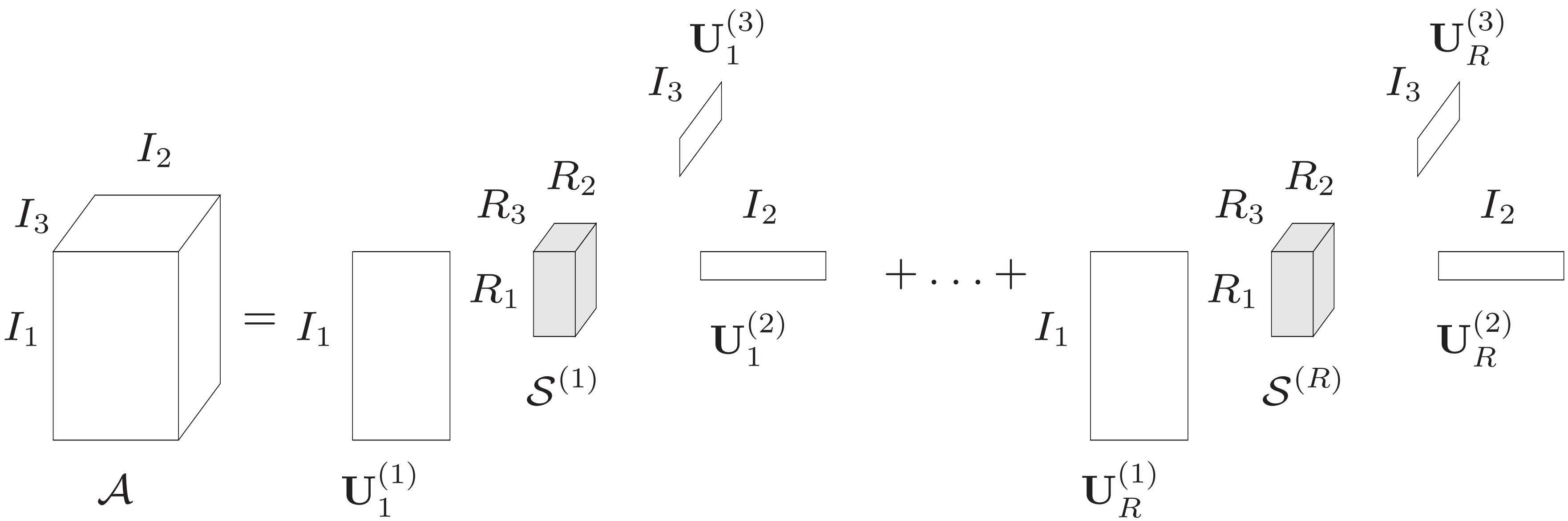
PARAFAC:



$$\mathbf{M}(\{U_r^{(2)}\}, \{U_r^{(3)}\})X = A$$

$$x(1 : I_1) \sim \lambda_1 U_1^{(1)}$$

Decomposition in rank-(R_1, R_2, R_3) terms:



$$\mathbf{M}(\{\mathbf{U}_r^{(2)}\}, \{\mathbf{U}_r^{(3)}\})X = A$$

$$x(1 : I_1 R_2 R_3) \quad \sim \quad \mathbf{U}_1^{(1)} \cdot \mathbf{S}^{(1)}$$

$$(I_1 \times R_1)(R_1 \times R_2 R_3)$$

Perspectives

- Simultaneous matrix decompositions
- Enhanced line search
- Levenberg-Marquardt
- Model order and model structure selection