Generic properties of Symmetric Tensors

Pierre COMON

I3S - CNRS

other contributors:

Bernard MOURRAIN

INRIA institute

Lek-Heng LIM,

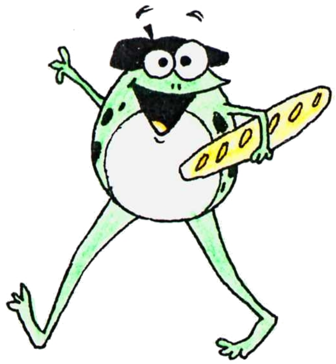
Stanford University

Tensors & Arrays

Definitions

Table $\boldsymbol{T} = \{T_{ij..k}\}$

- *Order* d of $\boldsymbol{T} \stackrel{\text{def}}{=} \#$ of its ways $= \#$ of its indices
- *Dimension* $n_\ell \stackrel{\text{def}}{=}$ range of the ℓth index
- $\boldsymbol{T}$ is *Square* when all dimensions $n_\ell = n$ are equal
- $\boldsymbol{T}$ is *Symmetric* when it is square and when its entries do not change by *any* permutation of indices



Tensors & Arrays

Properties

- Outer (tensor) product $\boldsymbol{C} = \boldsymbol{A} \circ \boldsymbol{B}$:

$$C_{ij..\ell\, ab..c} = A_{ij..\ell}\, B_{ab..c}$$

EXAMPLE 1 outer product between 2 vectors: $\boldsymbol{u} \circ \boldsymbol{v} = \boldsymbol{u}\,\boldsymbol{v}^{\mathsf{T}}$

- Multilinearity. An order-3 tensor $\boldsymbol{T}$ is transformed by the multi-linear map $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ into a tensor $\boldsymbol{T}'$:

$$T'_{ijk} = \sum_{abc} A_{ia} B_{jb} C_{kc} T_{abc}$$

Similarly: at any order d.

Tensors & Arrays

Example

EXAMPLE 2

Take

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then

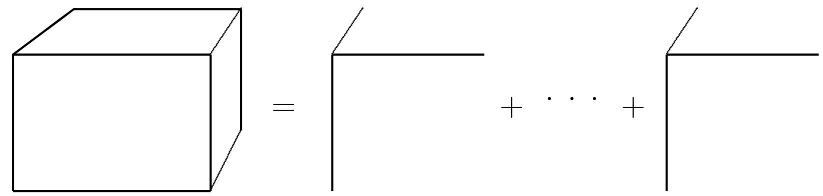
$$\boldsymbol{v}^{\circ 3} = \left(\begin{array}{rr|rr} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array}\right)$$

This is a “rank-1” symmetric tensor

Usefulness of symmetric arrays

CanD/PARAFAC vs ICA

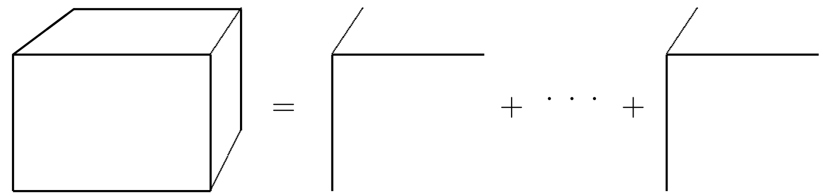
- **CanD/PARAFAC:**



Usefulness of symmetric arrays

CanD/PARAFAC vs ICA

- **CanD/PARAFAC:**

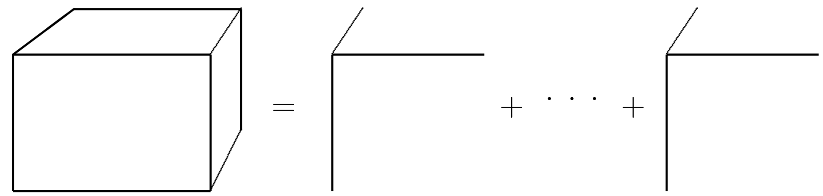


PARAFAC cannot be used when:

- Lack of diversity

Usefulness of symmetric arrays

CanD/PARAFAC vs ICA

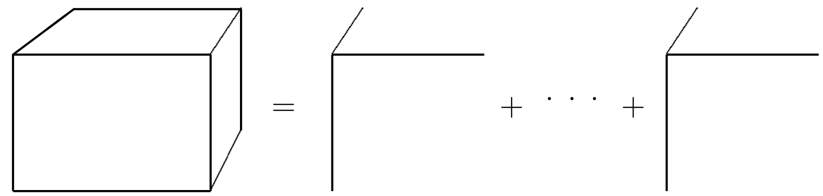
- **CanD/PARAFAC:** 

PARAFAC cannot be used when:

- Lack of diversity
- Proportional slices

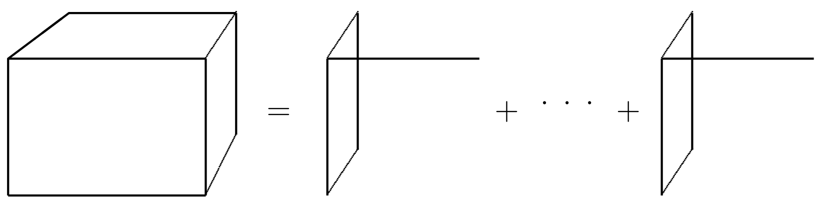
Usefulness of symmetric arrays

CanD/PARAFAC vs ICA

- **CanD/PARAFAC:** 

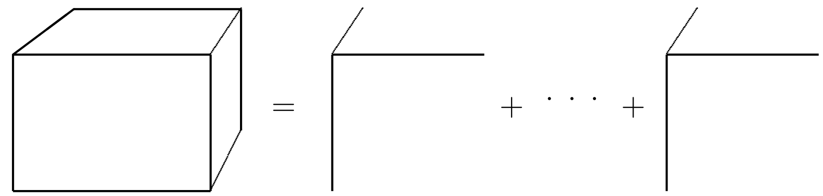
PARAFAC cannot be used when:

- Lack of diversity
- Proportional slices
- Lack of physical meaning (e.g.video)



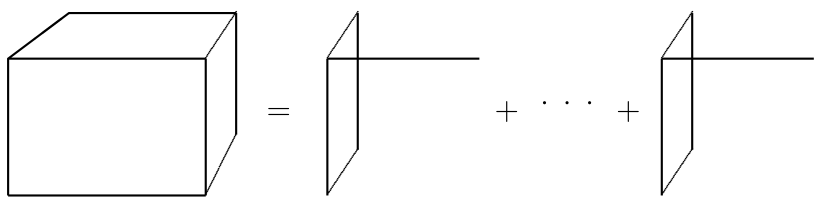
Usefulness of symmetric arrays

CanD/PARAFAC vs ICA

- **CanD/PARAFAC:** 

  PARAFAC cannot be used when:

  - Lack of diversity
  - Proportional slices
  - Lack of physical meaning (e.g.video)

- Then use *Independent Component Analysis* (ICA) [Comon'1991]

  ICA: decompose a *cumulant tensor* instead of the data tensor

Usefulness of symmetric arrays

Independent Component Analysis (ICA)

Advantages of ICA

- One can obtain a tensor of *arbitrarily large* order from a single data *matrix*.

Drawbacks of ICA

- One dimension of the *data matrix* must be much larger than the other
- Additional computational cost of the Cumulant tensor

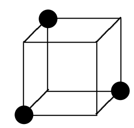
Tensors and Polynomials

Bijection

EXAMPLE 6 $(d, n) = (3, 2)$

$$p(x_1, x_2) = \sum_{i,j,k=1}^{2} T_{ijk}\, x_i\, x_j\, x_k$$

$$\boldsymbol{T} = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) =$$



$$\Rightarrow \;\; p(\boldsymbol{x}) = 3\, x_1^2 x_2 = 3\, \boldsymbol{x}^{[2,1]}$$

Tensors and Polynomials

Bijection

- Symmetric tensor of order d and dimension n can be associated with a unique homogeneous polynomial of degree d in n variables:

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{j}} T_{\boldsymbol{j}}\, \boldsymbol{x}^{\boldsymbol{f}(\boldsymbol{j})} \tag{1}$$

  - integer vector $\boldsymbol{j}$ of dimension $d \leftrightarrow$ integer vector $\boldsymbol{f}(\boldsymbol{j})$ of dimension n
  - entry f_k of $\boldsymbol{f}(\boldsymbol{j})$ being $\stackrel{\text{def}}{=}$ #of times index k appears in $\boldsymbol{j}$
  - We have in particular $|\boldsymbol{f}(\boldsymbol{j}))| = d$.

- Standard conventions $\boldsymbol{x}^{\boldsymbol{j}} \stackrel{\text{def}}{=} \prod_{k=1}^{n} x_k^{j_k}$ and $|\boldsymbol{f}| \stackrel{\text{def}}{=} \sum_{k=1}^{n} f_k$, where $\boldsymbol{j}$ and $\boldsymbol{f}$ are integer vectors.

Orbits

Definition

- General Linear group $\mathcal{GL}$: group of invertible matrices
- Orbit of a polynomial p: all polynomials q that can be transformed into p by $\boldsymbol{A} \in \mathcal{GL}$: $q(\boldsymbol{x}) = p(\boldsymbol{A}\boldsymbol{x})$.
- Allows to classify polynomials

Quadrics

quadratic homogeneous polynomials

- Binary quadrics (2×2 symmetric matrices)
  - Orbits in $\mathbb{R}$: $\{0,\ x^2,\ x^2+y^2,\ x^2-y^2\}$
    ☞ $2xy \in \mathcal{O}(x^2-y^2)$ in $\mathbb{R}[x,y]$
  - Orbits in $\mathbb{C}$: $\{0,\ x^2,\ x^2+y^2\}$
    ☞ $2xy \in \mathcal{O}(x^2+y^2)$ in $\mathbb{C}[x,y]$
- Set of singular matrices is closed
- Set $\mathcal{Y}_r$ of matrices of at most rank r is closed

Spaces of tensors

dimensions

- $\mathcal{A}_n$: square asymmetric of dimensions n and order d
  ☞ dimension $D_{_A}(n,d) = n^d$

Spaces of tensors

dimensions

- $\mathcal{A}_n$: square asymmetric of dimensions n and order d

  ☞ dimension $D_A(n,d) = n^d$

- $\mathcal{S}_n$: square symmetric of dimensions n and order d

  ☞ dimension $D_S(n,d) = \binom{n+d-1}{d}$

n\d	quadric 2	cubic 3	quartic 4	quintic 5	sextic 6
2	3	4	5	6	7
3	6	10	15	21	28
4	10	20	35	56	84
5	15	35	70	126	210
6	21	56	126	252	462

Number of free parameters in a symmetric tensor as a function of order d and dimension n

Definition of Rank

CAND

- Any tensor can always be decomposed (possibly non uniquely) as:

$$\boldsymbol{T} = \sum_{i=1}^{r} \boldsymbol{u}(i) \circ \boldsymbol{v}(i) \circ \ldots \boldsymbol{w}(i) \tag{2}$$

- DEFINITION
  Tensor rank $\stackrel{\text{def}}{=}$ minimal # of terms necessary

- This *Canonical decomposition* (CAND) holds valid in a *ring*

- The CAND of a multilinear transform = the multilinear transform of the CAND:
  - If $\boldsymbol{T} \xrightarrow{\mathcal{L}} \boldsymbol{T}'$ by a multilinear transform $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$,
  - then $(\boldsymbol{u}, \boldsymbol{v}, ..\boldsymbol{w}) \xrightarrow{\mathcal{L}} (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{B}\boldsymbol{v}, ..\boldsymbol{C}\boldsymbol{w})$

Ranks are difficult to evaluate

Clebsch theorem

Alfred Clebsch (1833-1872)

The generic ternary quartic cannot in general be written as the sum of 5 fourth powers

- $D(3, 4) = 15$
- $3\,r$ free parameters in the CAND
- But $r = 5$ is not enough $\rightarrow r = 6$ is generic

Questions

1. Rank vs Symmetric rank in $\mathcal{S}_n$

2. Generic rank, Typical rank
Differences between $\mathcal{S}_n$ and $\mathcal{A}_n$

3. Rank and CAND of a given tensor
Uniqueness
Closeness of sets of given rank

4. Maximal rank in $\mathcal{S}_n$ or $\mathcal{A}_n$

5. Differences between $\mathbb{R}$ and $\mathbb{C}$

Symmetric rank vs rank

given a symmetric tensor, $\boldsymbol{T}$, one can decompose it as

- a sum of *symmetric* rank-1 tensors
- a sum of rank-1 tensors

➨ Is the rank the same?

LEMMA $rank(\boldsymbol{T}) \leq rank_S(\boldsymbol{T})$

Symmetric CAND vs CAND

- Let $\boldsymbol{T} \in \mathcal{S}$ symmetric tensor, and its CAND:

$$\boldsymbol{T} = \sum_{k=1}^{r} \boldsymbol{T}_k$$

where $\boldsymbol{T}_k$ are rank-1.

- PROPOSITION 1

If the constraint $\boldsymbol{T}_k \in \mathcal{S}$ is relaxed, then the rank is still the same

- But $\boldsymbol{T}_k$'s need not be each symmetric when solution is not essentially unique

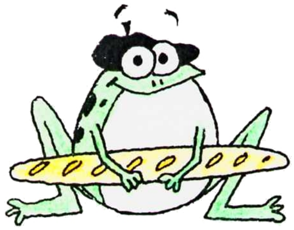
Proof. Generically when rank $\leq$ dimension, Always in dimension 2

Topology of polynomials

definition

- Every elementary closed set $\stackrel{\text{def}}{=}$ varieties, defined by $p(\boldsymbol{x}) = 0$
- Closed sets = finite union of varieties
- Closure of a set $\mathcal{E}$: smallest closed set $\overline{\mathcal{E}}$ containing $\mathcal{E}$

- called Zariski topology in $\mathbb{C}$
- this is not Euclidian topology, but results still apply



Tensor subsets

- Set of tensors of rank *at most r* with values in $\mathbb{C}$:

$$\mathcal{Y}_r = \{\boldsymbol{T} \in \mathcal{T} :\ r(\boldsymbol{T}) \leq r\}$$

- Set of tensors of rank *exactly r*: $\mathcal{Z}_r = \{\boldsymbol{T} \in \mathcal{T} :\ r(\boldsymbol{T}) = r\}$

$$\mathcal{Z} = \mathcal{Y}_r - \mathcal{Y}_{r-1},\ r > 1$$

- Zariski closures: $\overline{\mathcal{Y}}_r$, $\overline{\mathcal{Z}}_r$.

- PROPOSITION 3
  $\mathcal{Z}_1$ is closed *but not* $\mathcal{Z}_r,\ r > 1$ (intuitively obvious)

  [Burgisser'97] [Strassen'83]

  EXAMPLE
  $\boldsymbol{T}_\varepsilon = \boldsymbol{T}_0 + \varepsilon\, \boldsymbol{y}^{\circ d},\ \ \boldsymbol{T}_0 \in \mathcal{Z}_{r-1}$

Lack of closeness of $\mathcal{Y}_r$

PROPOSITION 4 If $d > 2$, $\mathcal{Y}_r$ is not closed for $1 < r < R$.

Proof. $\exists$ Sequence of rank-2 tensors converging towards a rank-4:

$$\boldsymbol{T}_\varepsilon = \frac{1}{\varepsilon} \left[(\boldsymbol{u} + \varepsilon\, \boldsymbol{v})^{\circ 4} - \boldsymbol{u}^{\circ 4}\right]$$

In fact, as $\varepsilon \longrightarrow 0$, it tends to:

$$\boldsymbol{T}_0 = \boldsymbol{u} \circ \boldsymbol{v} \circ \boldsymbol{v} \circ \boldsymbol{v} + \boldsymbol{v} \circ \boldsymbol{u} \circ \boldsymbol{v} \circ \boldsymbol{v} + \boldsymbol{v} \circ \boldsymbol{v} \circ \boldsymbol{u} \circ \boldsymbol{v} + \boldsymbol{v} \circ \boldsymbol{v} \circ \boldsymbol{v} \circ \boldsymbol{u}$$

which can be shown to be proportional to the rank-4 tensor (3).

Maximal rank

Example

EXAMPLE 3

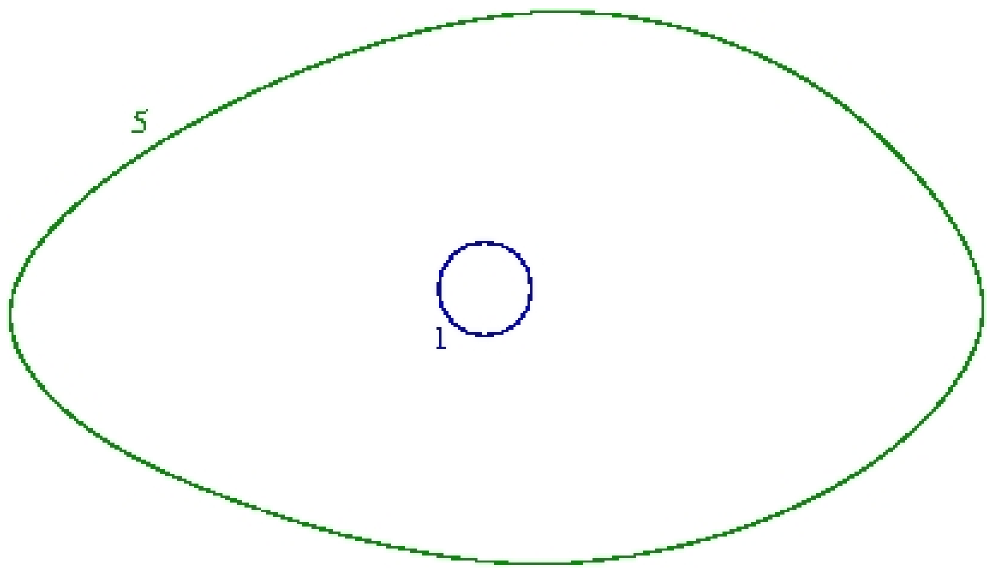
Tensor of dimension 2 and rank 4:

$$\boldsymbol{T} = 8\,(\boldsymbol{u}+\boldsymbol{v})^{\circ 4} - \;8(\boldsymbol{u}-\boldsymbol{v})^{\circ 4} - (\boldsymbol{u}+2\boldsymbol{v})^{\circ 4} + (\boldsymbol{u}-2\boldsymbol{v})^{\circ 4} \qquad (3)$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are not collinear

Lack of closeness of $\mathcal{Y}_r$

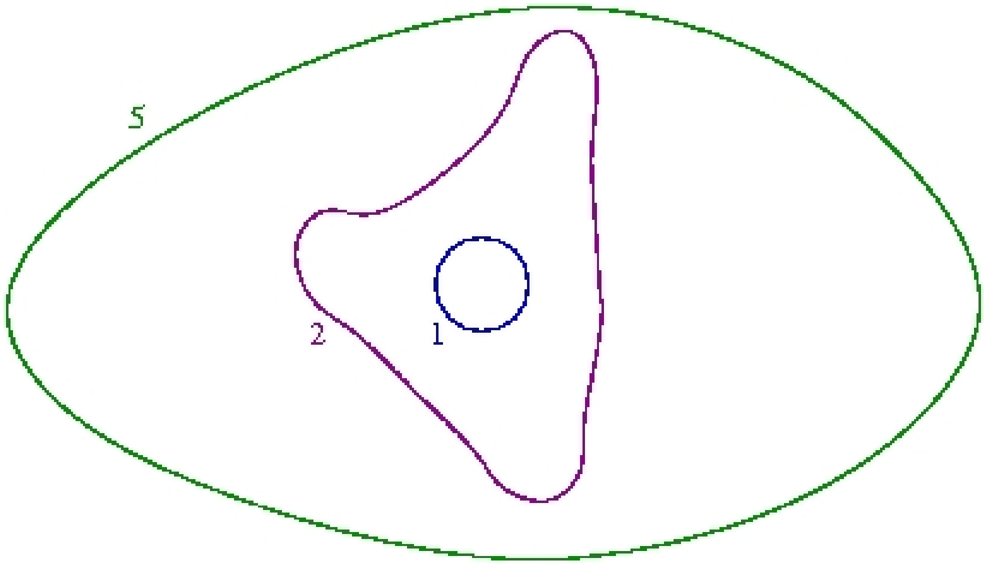
Successive sets $\mathcal{Y}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) \leq r\}$



- A tensor sequence in $\mathcal{Y}_r$ can converge to a limit in $\mathcal{Y}_{r+h}$

Lack of closeness of $\mathcal{Y}_r$

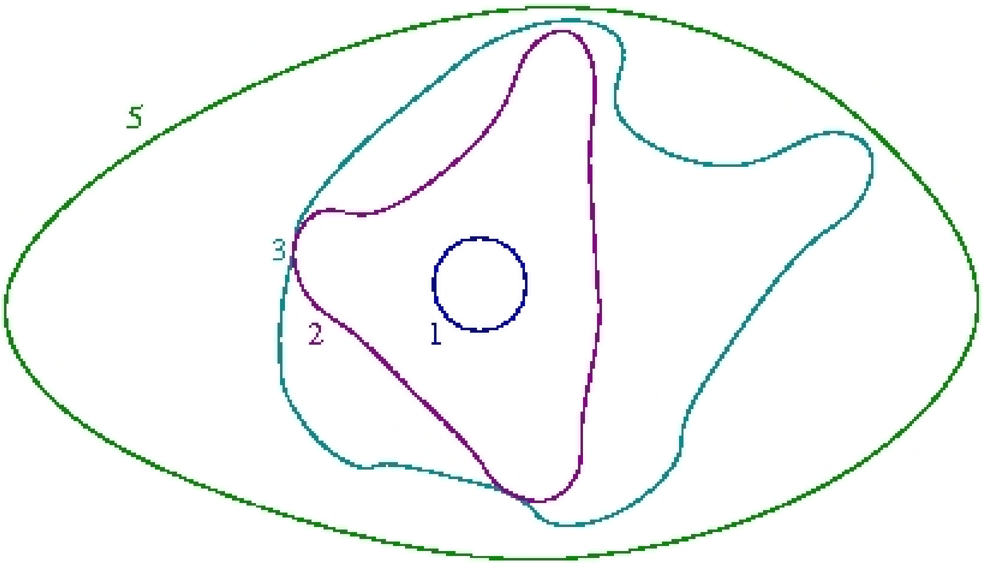
Successive sets $\mathcal{Y}_r = \{\boldsymbol{T} : \ rank(\boldsymbol{T}) \leq r\}$



- A tensor sequence in $\mathcal{Y}_r$ can converge to a limit in $\mathcal{Y}_{r+h}$

Lack of closeness of $\mathcal{Y}_r$

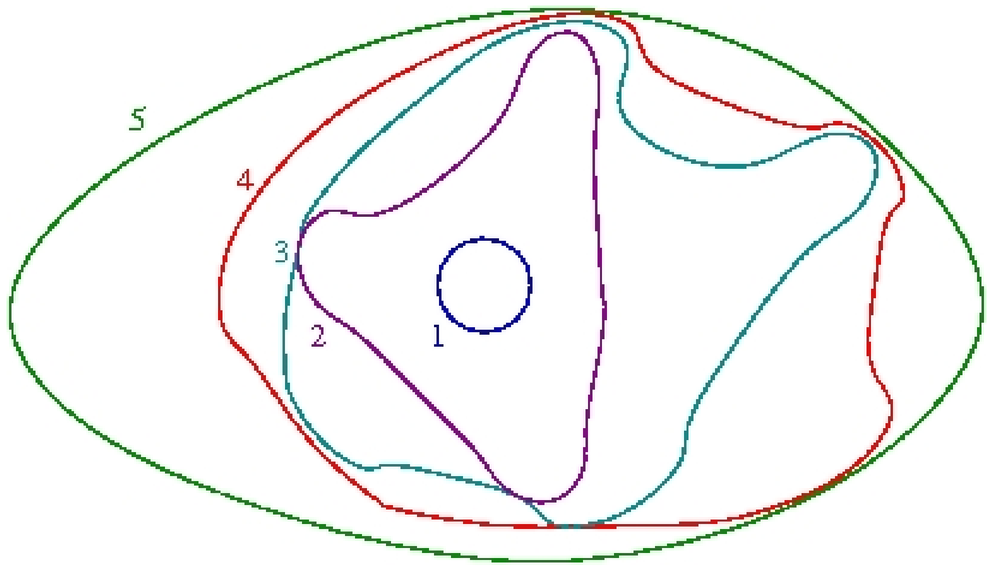
Successive sets $\mathcal{Y}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) \leq r\}$



- A tensor sequence in $\mathcal{Y}_r$ can converge to a limit in $\mathcal{Y}_{r+h}$

Lack of closeness of $\mathcal{Y}_r$

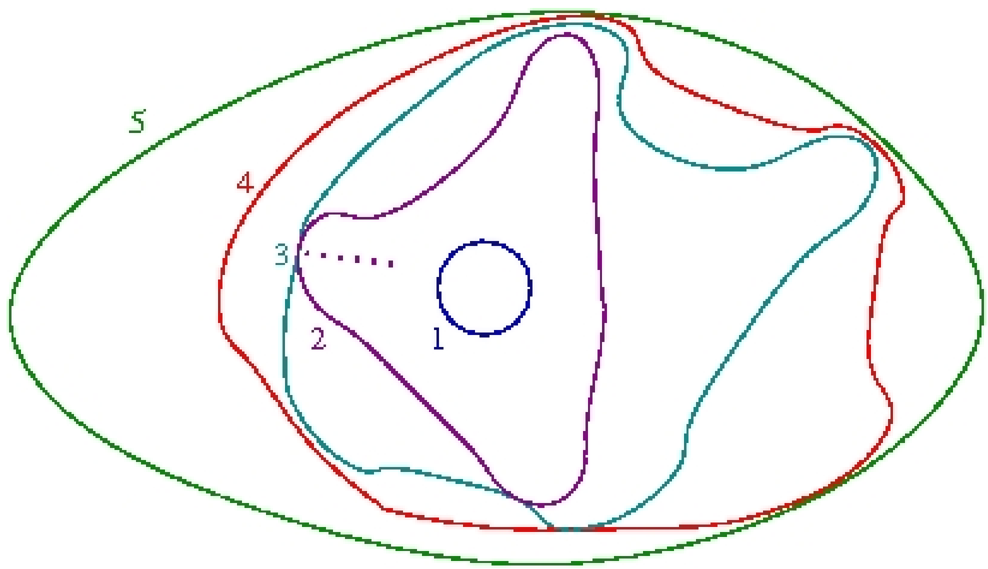
Successive sets $\mathcal{Y}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) \leq r\}$



- A tensor sequence in $\mathcal{Y}_r$ can converge to a limit in $\mathcal{Y}_{r+h}$

Lack of closeness of $\mathcal{Y}_r$

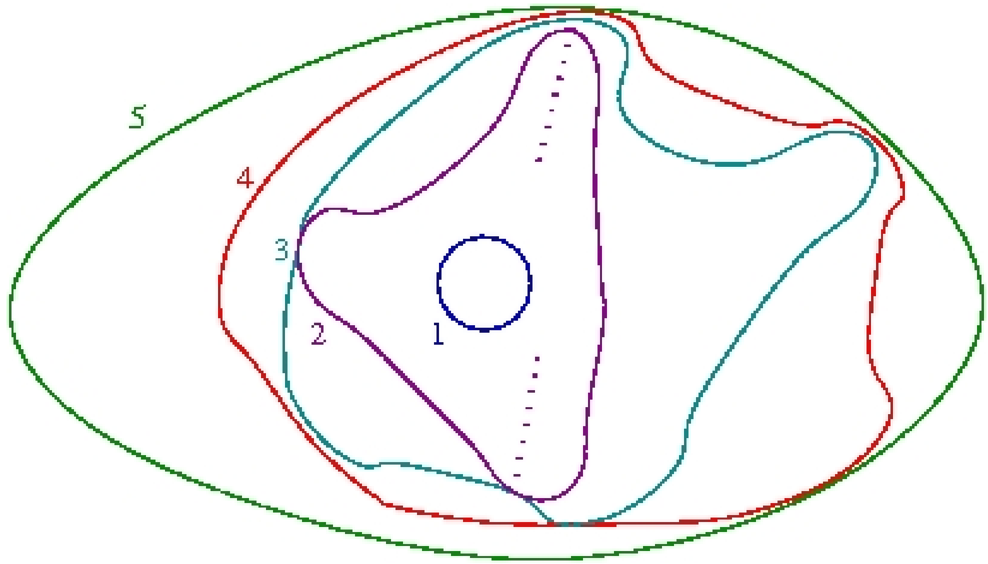
Successive sets $\mathcal{Y}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) \leq r\}$



- A tensor sequence in $\mathcal{Y}_r$ can converge to a limit in $\mathcal{Y}_{r+h}$

Lack of closeness of $\mathcal{Y}_r$

Successive sets $\mathcal{Y}_r = \{\boldsymbol{T} : \; rank(\boldsymbol{T}) \leq r\}$



- A tensor sequence in $\mathcal{Y}_r$ can converge to a limit in $\mathcal{Y}_{r+h}$

Why is it possible?

PROPOSITION 2

Given a set of vectors $\boldsymbol{u}[i] \in \mathbb{C}^N$ that are not pairwise collinear, there exists some integer d such that $\{\boldsymbol{u}^{\circ d}\}$ are *linearly independent*.

Related results in [Sidi-Bro-J.Chemo'2000]

Nullstellensatz

Hilbert's zero theorem

David Hilbert (1862-1943)

Genericity

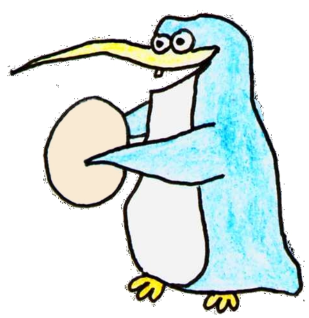
Intuitive

- A property is *typical* ⇔ is is true on a non zero volume set
- A property is *generic* ⇔ is is true almost everywhere

Mathematical

- DEFINITION r is a typical rank if (density argument with Zariski):
  $\overline{\mathcal{Z}}_r$ *is the whole space*

- DEFINITION Generic rank is *the typical rank when unique*



Generic rank in $\mathbb{C}$

Existence

- LEMMA (in either $\mathbb{R}$ of $\mathbb{C}$, either symmetric or not)
Strictly increasing series of $\overline{\mathcal{Y}}_k$ for $k \leq \overline{R}$, then constant:

$$\overline{\mathcal{Y}}_1 \subset_{\neq} \overline{\mathcal{Y}}_2 \subset_{\neq} \dots \subset_{\neq} \overline{\mathcal{Y}}_{\overline{R}} = \overline{\mathcal{Y}}_{\overline{R}+1} = \dots \mathcal{T}$$

which guarantees the existence of a unique $\overline{R}$

- PROPOSITION 5 For tensors in $\mathbb{C}$
If $r_1 < r_2 < \overline{R}$, then

$$\overline{\mathcal{Z}}_{r_1} \subset \overline{\mathcal{Z}}_{r_2} \subset \overline{\mathcal{Z}}_{\overline{R}} \tag{4}$$

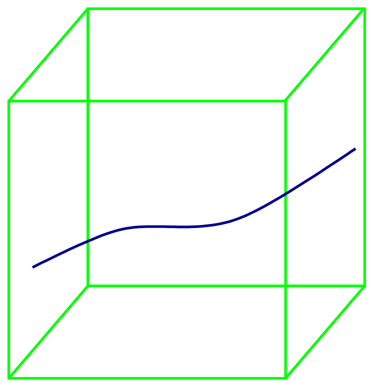
- PROPOSITION 6 For tensors in $\mathbb{C}$
If $\overline{R} < r_3 \leq R$, then

$$\overline{\mathcal{Z}}_{\overline{R}} \supset \overline{\mathcal{Z}}_{r_3} \supseteq \overline{\mathcal{Z}}_R$$

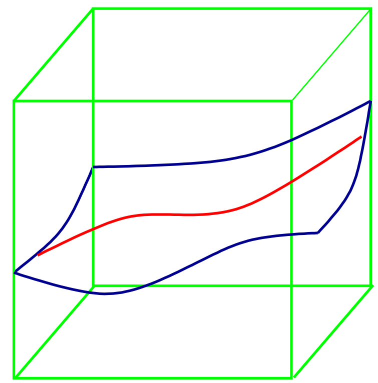
➤ Prove that $\overline{R}$ is the generic rank in $\mathbb{C}$

Generic rank

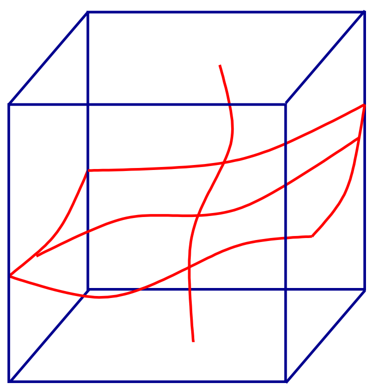
e.g. binary quartics in $\mathbb{C}$



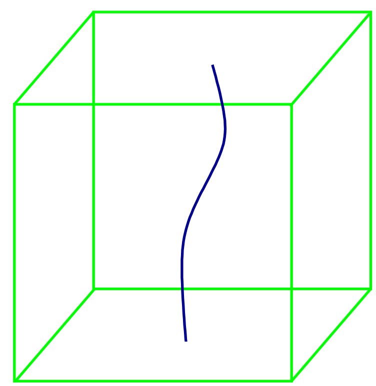
$\mathcal{Z}_1$



$\mathcal{Z}_2 = \mathcal{Y}_2 - \mathcal{Z}_1$



$$\mathcal{Z}_3 = \mathcal{Y}_3 - \mathcal{Z}_1 - \mathcal{Z}_2$$
$$= \mathcal{T} - \mathcal{Z}_1 - \mathcal{Z}_2 - \mathcal{Z}_4$$



$\mathcal{Z}_4 = \mathcal{Y}_4 - \mathcal{Y}_3$

Generic rank in $\mathbb{C}$

Computation based on a mapping

Symmetric

$$\{\boldsymbol{u}(\ell),\ 1 \leq \ell \leq r\} \xrightarrow{\varphi} \sum_{\ell=1}^{r} \boldsymbol{u}(\ell)^{\circ d}$$

$$\{\mathbb{C}^n\}^r \xrightarrow{\varphi} \mathcal{S}$$

Asymmetric

$$\{\boldsymbol{u}(\ell), \boldsymbol{v}(\ell), \dots, \boldsymbol{w}(\ell),\ 1 \leq \ell \leq r\} \xrightarrow{\varphi} \sum_{\ell=1}^{r} \boldsymbol{u}(\ell) \circ \boldsymbol{v}(\ell) \circ \dots \circ \boldsymbol{w}(\ell)$$

$$\{\mathbb{C}^{n_1} \circ \dots \circ \mathbb{C}^{n_d}\}^r \xrightarrow{\varphi} \mathcal{A}$$

Rank

The rank of the Jacobian of φ equals $\dim(\bar{\mathcal{Z}}_r)$, and hence D for large enough r.

➨ The smallest r for wich $rank(\text{Jacobian}(\varphi)) = D$ is $\bar{R}$.

Generic rank in $\mathbb{C}$

Example of computation

$$\{\boldsymbol{a}(\ell), \boldsymbol{b}(\ell), \boldsymbol{c}(\ell)\} \stackrel{\varphi}{\longrightarrow} \boldsymbol{T} = \sum_{\ell=1}^{r} \boldsymbol{a}(\ell) \circ \boldsymbol{b}(\ell) \circ \boldsymbol{c}(\ell)$$

In the canonical basis, $\boldsymbol{T}$ has coordinate vector:

$$\sum_{\ell=1}^{r} \boldsymbol{a}(\ell) \otimes \boldsymbol{b}(\ell) \otimes \boldsymbol{c}(\ell)$$

Hence the Jacobian of φ is the $r(n_1 + n_2 + n_3) \times n_1 n_2 n_3$ matrix:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I}_{n_1} & \otimes & \boldsymbol{b}^{\mathsf{T}}(1) & \otimes & \boldsymbol{c}^{\mathsf{T}}(1) \\ \boldsymbol{I}_{n_1} & \otimes & \dots & \otimes & \dots \\ \boldsymbol{I}_{n_1} & \otimes & \boldsymbol{b}^{\mathsf{T}}(r) & \otimes & \boldsymbol{c}^{\mathsf{T}}(r) \\ \boldsymbol{a}(1)^{\mathsf{T}} & \otimes & \boldsymbol{I}_{n_2} & \otimes & \boldsymbol{c}^{\mathsf{T}}(1) \\ \dots & \otimes & \boldsymbol{I}_{n_2} & \otimes & \dots \\ \boldsymbol{a}(r)^{\mathsf{T}} & \otimes & \boldsymbol{I}_{n_2} & \otimes & \boldsymbol{c}^{\mathsf{T}}(r) \\ \boldsymbol{a}(1)^{\mathsf{T}} & \otimes & \boldsymbol{b}(1)^{\mathsf{T}} & \otimes & \boldsymbol{I}_{n_3} \\ \dots & \otimes & \dots & \otimes & \boldsymbol{I}_{n_3} \\ \boldsymbol{a}(r)^{\mathsf{T}} & \otimes & \boldsymbol{b}(r)^{\mathsf{T}} & \otimes & \boldsymbol{I}_{n_3} \end{bmatrix}$$

$rank(\boldsymbol{J}) = dim(\mathrm{Im}(\varphi))$ and $\bar{R} = Min\{r : \mathrm{Im}\{\varphi\} = \mathcal{A}\}$

Generic rank $\bar{R}(d,n)$ in $\mathbb{C}$

Results

Symmetric

d \ n	2	3	4	5	6
3	2	4	5	**8**	10
4	3	**6**	**10**	**15**	21

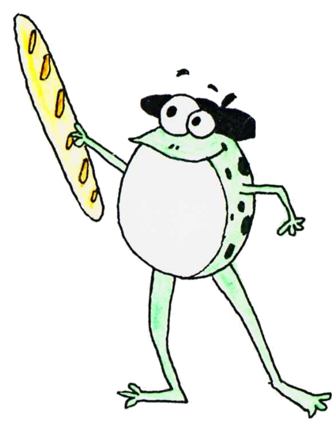
$$\bar{R} \geq \frac{1}{n}\binom{n+d-1}{d}$$

Asymmetric

d \ n	2	3	4	5	6
3	2	**5**	7	10	14
4	4	9	20	37	62

$$\bar{R} \geq \frac{n^d}{nd-d+1}$$

bold: exceptions to the ceil rule



Exceptions to the ceil rule

THEOREM For $d > 2$, the generic rank of a dth order symmetric tensor of dimension n is always equal to the lower bound

$$\bar{R}_s = \left\lceil \frac{\binom{n+d-1}{d}}{n} \right\rceil \qquad (5)$$

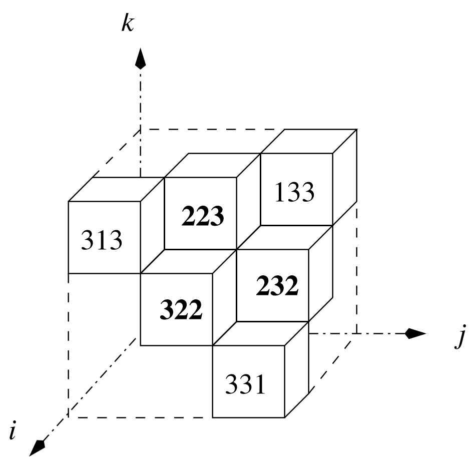
except for the following cases: $(d, n) \in \{(3,5), (4,3), (4,4), (4,5)\}$, for which it should be increased by 1.

Proof see Alexander-Hirschowitz theorem on multivariate interpolation (cf. appendix).

Classification of ternary cubics

$3 \times 3 \times 3$

$\mathcal{GI}-$orbit	$\omega(p)$
x^3	1
$x^2y + xy^2$	2
x^2y	3
$x^3 + 3\,y^2z$	4
$x^3 + y^3 + 6\,xyz$	4
$x^3 + 6\,xyz$	4
$a\,(x^3 + y^3 + z^3) + 6b\,xyz$	4 (**generic**)
$xz^2 + y^2z$	5



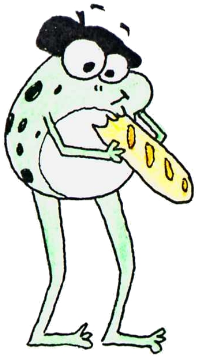
Maximal rank

George Salmon (1819-1904)

Dimension of solutions

Calculation

- **Asymmetric $d = 3$**
  $F(n_1, n_2, n_3) = (n_1 + n_2 + n_3 - 2)\,\bar{R} - n_1\,n_2\,n_3$
- **Asymmetric square**
  $F(n) = (nd - d + 1)\bar{R} - n^d$
- **Symmetric**
  $F(n) = n\,\bar{R} - \binom{n+d-1}{d}$



Dimension of solutions

Uniqueness

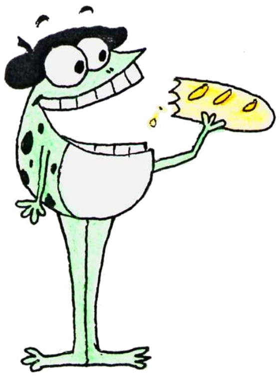
Symmetric

d \ n	2	3	4	5	6
3	0	2	0	**5**	4
4	1	**3**	**5**	**5**	0

Asymmetric

d \ n	2	3	4	5	6
3	0	**8**	6	5	8
4	4	0	4	4	6

- Insights on uniqueness



Dimension of solutions

General tool

General tool to assess uniqueness

EXAMPLE Indscal:

$$\boldsymbol{X}_{ijk} = \sum_{\ell} A_{i\ell}\, A_{j\ell}\, C_{k\ell}$$

EXAMPLE Parafac2:

$$\boldsymbol{X}_{ijk} = \sum_{\ell} A_{i\ell}\, B_{j\ell}\, C_{jk\ell}$$

with constraints

EXAMPLE :

$$\boldsymbol{X}_{ijk} = \sum_{\ell} A_{i\ell}\, B_{j\ell}\, C_{jk\ell}\, D_{ik\ell}$$

with constraints

Typical ranks in $\mathbb{R}$

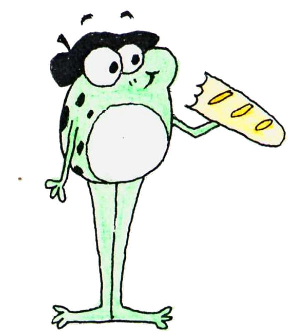
Lack of uniqueness in $\mathbb{R}$

- Draw randomly entries of a tensor $\in \mathcal{T}(n, d)$ according to a distribution $q(t)$
- Typical ranks do not depend on $q(t)$, if c.d.f. absolutely continuous (no point-like mass). Only volumes of Z_r do.
- Typical ranks depend on (n, d)

EXAMPLE 4 $2 \times 2 \times 2$ asymmetric tensors

- drawn according to Gaussian symmetric $\Rightarrow \{2(57\%), 3(43\%)\}$
- drawn according to Gaussian asymmetric $\Rightarrow \{2(80\%), 3(20\%)\}$

More on this matter: [ten Berge] [Stegeman]



Ranks in $\mathbb{R}$

vs rank in $\mathbb{C}$

- $\forall \boldsymbol{T}$ real tensor, rank in $\mathbb{R}$ always larger than rank in $\mathbb{C}$:

$$rank^{\mathbb{C}}(\boldsymbol{T}) \leq rank^{\mathbb{R}}(\boldsymbol{T})$$

- In particular:

$$\text{generic rank} \leq \text{typical ranks}$$

Example 5

$$\boldsymbol{T}(:,:,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{T}(:,:,2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

- If decomposed in $\mathbb{R}$, it is of rank 3:

$$\boldsymbol{T} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\circ 3} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\circ 3} - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\circ 3}$$

- whereas it admits a CAND of rank 2 in $\mathbb{C}$:

$$\boldsymbol{T} = \frac{\jmath}{2}\begin{pmatrix} -\jmath \\ 1 \end{pmatrix}^{\circ 3} - \frac{\jmath}{2}\begin{pmatrix} \jmath \\ 1 \end{pmatrix}^{\circ 3}$$

Future works

Results

- Generic rank *unique and known* in CanD for every (d, n)
- Rank = Symmetric Rank *(partial)*
- $\mathcal{Y}_r$ is never closed for $1 < r < R$ and $2 < d$ *(partial)*
- Generic rank can be computed for any $d-$way model
- $\mathbb{C}$ easier than $\mathbb{R}$: some hope to have more general results

Open questions

- *Maximal* achievable ranks as a function of (d, n)?
- What does "*low-rank approximation*" means for tensors when rank> 1?
- Only 2 *typical* ranks may exist for $\mathbb{R}$ tensors?

APPENDIX

Dimension 2: Sylvester

2x2x...x2

Sylvester's theorem in $\mathbb{C}$

A binary quantic $p(x,y) = \sum_{i=0}^{d} c(i)\,\gamma_i\,x^i\,y^{d-i}$ can be written as a sum of dth powers of r distinct linear forms:

$$p(x,y) = \sum_{j=1}^{r} \lambda_j\,(\alpha_j\,x + \beta_j\,y)^d, \tag{6}$$

if and only if **(i)** there exists a vector $\boldsymbol{g}$ of dimension $r+1$, with components g_ℓ, such that

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r \\ \vdots & & & \vdots \\ \gamma_{d-r} & \cdots & \gamma_{d-1} & \gamma_d \end{bmatrix} \boldsymbol{g}^* = \mathbf{0}. \tag{7}$$

and **(ii)** the polynomial $q(x,y) \stackrel{\text{def}}{=} \sum_{\ell=0}^{r} g_\ell\,x^\ell\,y^{r-\ell}$ admits r distinct roots

Alexander-Hirschowitz Theorem

Polynomial interpolation

Let $\mathcal{L}(d,m)$ be the space of hypersurfaces of degree at most d in m variables. This space is of dimension $D(m,d) \stackrel{\text{def}}{=} \binom{m+d}{d} - 1$.

THEOREM Denote $\{p_i\}$ n given distinct points in the complex projective space $\mathbb{P}^m$. The dimension of the linear subspace of hypersurfaces of $\mathcal{L}(d,m)$ having multiplicity at least 2 at every point p_i is:

$$D(m,d) - n(m+1)$$

except for the following cases:

- $d = 2$ and $2 \le n \le m$
- $d \ge 3$ and $(m,d,n) \in \{(2,4,5),(3,4,9),(4,1,14),(4,3,7)\}$

In other words, there are a *finite number* of exceptions.